Problem

Consider the equation

$$dw = \Omega, \qquad \deg \Omega = k + 1$$

(a)

Prove that solution exists if and only if Ω is closed and homology class $[\Omega] \in H^{k+1}(M)$ equals to zero.

(b)

Prove that for any two solutions holds

$$d(w - w') = 0$$

and the set of all solutions is a coset of w by subspace of all closed forms of degree k.

(c)

Prove space of all closed forms of degree k is isomorphic to direct sum of space of exact forms of degree k and cohomology group $H^k(M)$.

Solution

If w is the solution of the equation then Ω is exact form, so $[\Omega] = 0$. Also, if $[\Omega] = 0$ then Ω is exact form, so $\Omega = dw$ for some form w that is the solution to equation.

Let w and w' are solutions of the equation $dw = \Omega$. Then

$$d(w - w') = dw - dw' = \Omega - \Omega = 0$$

so $w - w'$ is closed form. So every solution w' may be received by adding to w some closed form. Let's denote by $\Omega_k(M)$ linear space of all outer differential forms of degree k on manifold M. Thus gradient d is linear map

$$d: \Omega_k(M) \to \Omega_{k+1}(M)$$

Space of closed forms of degree k equals to kernel $Ker\ d \subset \Omega_k(M)$ of d. In the same way gradient d maps the space $\Omega_{k-1}(M)$ to $\Omega_k(M)$:

$$d: \Omega_{k-1}(M) \to \Omega_k(M)$$

Then the space of exact form of degree k equals to image of the map $Im\ d = d\big(\Omega_{k-1}(M)\big) \subset Ker\ d \subset \Omega_k(M)$. So the group of k-dimensional cohomologies $H^k(M)$ equals to the factor-space $Ker\ d/\ Im\ d$. Consider algebraic completion H' to subspace $Im\ d \subset Ker\ d$. Then $Ker\ d = Im\ d \oplus H'$.

Let's show H' is isomorphic to group of cohomologies $H^k(M)$. Let $\phi: H' \to H^k(M)$ is the map that returns adjacency class $[w] \in H^k(M)$. If $\phi(w) = 0$ then $[w] = 0$ so w is exact form that implies $w \in Im\ d$. Since $Im\ d \cap H'$ contains only zero element then $w = 0$. So ϕ is monomorphism. Let $x \in H^k(M)$ is arbitrary element. Then x is adjacency class of some form $w \in Ker\ d$ by subspace $Im\ d, x = [w]$. So $w = d\Omega + w', d\Omega \in Im\ d, w' \in H'$. Then $x = [w] = [w;] = \phi(w')$. So ϕ is epimorphism. So we proved ϕ is isomorphism.