Problem 1

Let random variable X is uniformly distributed on [a;b], where a and b are unknown parameters. Find the maximal likelihood estimation of a and b.

Solution

To solve the problem at first we have to find density (or probabilities in discrete case) of random variable.

Density of uniform distribution is:

$$f(x,a,b) = \begin{cases} \frac{1}{b-a}, a \leq x \leq b \\ 0, otherwise \end{cases}$$

Likelihood function is product of densities in sample points.

Likelihood in this case is:

$$L(x_1, x_2, \dots x_n, a, b) = \prod_{k=1}^{k=n} f(x_k, a, b) = \left(\frac{1}{b-a}\right)^n$$

if $a \leq x_k \leq b$ for all and 0 otherwise. So, likelihood function is not equal to 0 if such inequalities holds:

$$b \geq x^{(n)} = \max_{k=1,2,\dots,n} x_k$$

$$a \leq x^{(1)} = \min_{k=1,2,\dots,n} x_k$$

Maximum likelihood estimation is value of unknown parameters that maximize likelihood function. Function $L(x_1, \dots, x_n, a, b)$ reaches its maximum when difference $(b-a)$ is minimal, but satisfying 2 inequalities above, that is when

$$b = \max_{k=1,2,\dots,n} x_k$$

$$a = \min_{k=1,2,\dots,n} x_k$$

This are maximal likelihood estimations for a and b.

Problem 2

Let random variable X has Poisson distribution with unknown parameter a. Find the maximal likelihood estimation for a.

Solution

Likelihood function is equal to:

$$L(x_1, \dots, x_n, a) = \prod_{k=1}^{k=n} \frac{a^{x_k} e^{-a}}{x_k!}$$

Logarithm of likelihood function is equal to:

$$\ln L(x_1, \dots, x_n, a) = \ln \prod_{k=1}^{k=n} \frac{a^{x_k} e^{-a}}{x_k!}$$
$$= (logarithm\ of\ product\ is\ equal\ to\ sum\ of\ logarithms)$$
$$= \sum_{k=1}^{k=n} \ln \frac{a^{x_k} e^{-a}}{x_k!} = \sum_{k=1}^{k=n} (\ln a^{x_k} + \ln e^{-a} - \ln x_k!)$$
$$= \left(\sum_{k=1}^{k=n} x_k \ln a - \ln x_k! \right) - na$$

Now we have to find maximum of this function on a. To do this let's find derivative of this function on a and equate it with 0 (if the function reaches maximum at some point, then it's derivative equals to 0 there):

$$\frac{\partial \ln L\,(x_1, \dots, x_n, a)}{\partial a} = \frac{1}{a} \left(\sum_{k=1}^{k=n} x_k \right) - n = 0$$

Now expressing a from equation we got we have:

$$a = \left(\sum_{k=1}^{k=n} x_k \right) / n$$

that is the answer to the problem.

Problem 3

Find maximum likelihood estimation of p – probability of success in Bernoulli trial.

Solution

It this case distribution is Bernoulli with unknown parameter p.

$$\tilde{f}(x,p) = \begin{cases} p, x = 1 \\ 1 - p, x = 0 \end{cases}$$

It is hard to work with such function, so we will change it to continuous and defined everywhere:

$$f(x,p) = p^x(1-p)^{1-x}$$

We see that

$$f(1,p) = p^1(1-p)^0 = p$$

$$f(0,p) = p^0(1-p)^1 = 1 - p$$

So, this function equals to $\tilde{f}(x,p)$ in every point $\tilde{f}(x,p)$ is defined. So, everything is done, and we may work with $f(x,p)$ further.

$$L(x_1, x_2, \dots, x_n, p) = \prod_{k=1}^{k=n} p^{x_k}(1-p)^{1-x_k}$$

$$\ln L(x_1, x_2, \dots, x_n, p) = \sum_{k=1}^{k=n} (x_k \ln p + (1 - x_k)\ln(1-p))$$

$$\frac{\partial}{\partial p} \ln L(x_1, x_2, \dots, x_n, p) = \sum_{k=1}^{k=n} \left(\frac{x_k}{p} + \frac{1 - x_k}{1 - p}\right) = 0$$

Solving this on parameter p we have:

$$\hat{p} = \frac{1}{n} \sum_{k=1}^{k=n} x_k$$

that is the maximal likelihood estimation for p.