Problem 1

Let T_i denote the time to the ith point in a Poisson process with rate λ on $[0, \infty)$.

(i) For fixed $0 < t_1 < t_2 < \cdots < t_k < \infty$ and a suitably small $\delta > 0$, calculate

$$P(t_1 \le T_1 < t_1 + \delta, t_2 \le T_2 < t_2 + \delta, \dots, t_k \le T_k < t_k + \delta)$$

up to an error that is smaller order than δ^k.

(ii) Find the joint density function for $(T_1, T_2, \dots, T_k)$

(iii) Show that conditional on $T_3 = t_3$ for a given $t_3 > 0$, the random variables $\left(\frac{T_1}{T_3}, \frac{T_2}{T_3}\right)$ are jointly uniformly distributed over the set

$$\{(u_1, u_2) \in \mathbb{R}^2 : 0 < u_1 < u_2 < 1\}$$

Solution

i)

Denote $\xi_{a,b}$ number of points of Poisson process during time $[a, b)$.

$\xi_{a,b}$ has Poisson distribution with parameter $\lambda(b-a)$: $P(\xi_{a,b} = k) = \frac{((b-a)\lambda)^k e^{-\lambda(b-a)}}{k!}$

$$P(t_1 \le T_1 < t_1 + \delta, t_2 \le T_2 < t_2 + \delta, \dots, t_k \le T_k < t_k + \delta)$$
$$= P\left(\xi_{0,t_1} = 0, \xi_{t_1,t_1+\delta} = 1, \xi_{t_1+\delta,t_2} = 0, \xi_{t_2,t_2+\delta} = 1, \dots, \xi_{t_{k-1}+\delta,t_k} = 0, \xi_{t_k,t_k+\delta} = 1\right)$$
$$= (numbers\ of\ points\ on\ disjoint\ intervals\ of\ Poisson\ process\ are\ independent) =$$

$$P\left(\xi_{0,t_1} = 0\right)P\left(\xi_{t_1,t_1+\delta} = 1\right)P\left(\xi_{t_1+\delta,t_2} = 0\right)P\left(\xi_{t_2,t_2+\delta} = 1\right) \cdot \dots \cdot P\left(\xi_{t_{k-1}+\delta,t_k} = 0\right)P\left(\xi_{t_k,t_k+\delta} = 1\right)$$
$$= e^{-\lambda t_1}(\delta\lambda)e^{-\delta\lambda}e^{-\lambda(t_2-t_1-\delta)}(\delta\lambda)e^{-\delta\lambda} \cdot \dots \cdot e^{-\lambda(t_k-t_{k-1}-\delta)}(\delta\lambda)e^{-\delta\lambda}$$
$$= \left((\delta\lambda)e^{-\delta\lambda}\right)^k e^{\lambda(\delta k - t_k)} = (\delta\lambda)^k e^{-\lambda t_k}$$

ii)

Joint density is

$$f_{T_1,T_2,\dots,T_k}(t_1, t_2, \dots t_n) = \lim_{\delta\to 0} \frac{P(t_1 \le T_1 < t_1 + \delta, t_2 \le T_2 < t_2 + \delta, \dots, t_k \le T_k < t_k + \delta)}{\delta^k}$$
$$= \lim_{\delta\to 0} \frac{(\delta\lambda)^k e^{-\lambda t_k}}{\delta^k} = \lambda^k e^{-\lambda t_k}$$

Finally,

$$f_{T_1,T_2,\dots,T_k}(t_1, t_2, \dots t_k) = \begin{cases} \lambda^k e^{-\lambda t_k}, 0 < t_1 < t_2 < \cdots < t_k \\ 0, othewise \end{cases}$$

iii)

Let's find distribution of $\left(\frac{T_1}{T_3}, \frac{T_2}{T_3}\right)$ $conditional\ T_3 = t_3$

$$F_{\frac{T_1}{T_3},\frac{T_2}{T_3}}(s,r) = P\left(\frac{T_1}{T_3} \le s, \frac{T_2}{T_3} \le r | T_3 = t_3\right) = P\left(T_1 < st_3, T_2 < rt_3 \middle| \xi_{0,t} = 2\right)$$

$$= \frac{P\left(T_1 < st_3, T_2 < rt_3, \xi_{0,t_3} = 2\right)}{P\left(\xi_{0,t} = 2\right)}$$

$$= \frac{\left(P\left(\xi_{0,st_3} = 1, \xi_{st_3,rt_3} = 1, \xi_{rt_3,t_3} = 0\right) + P\left(\xi_{0,st_3} = 2, \xi_{st_3,t_3} = 0\right)\right)}{P\left(\xi_{0,t_3} = 2\right)}$$

$$= (using\ independency)$$

$$= \frac{P\left(\xi_{0,st_3} = 1\right)P\left(\xi_{st_3,rt_3} = 1\right)P\left(\xi_{rt_3,t_3} = 0\right) + P\left(\xi_{0,st_3} = 2\right)P\left(\xi_{st_3,t_3} = 0\right)}{P\left(\xi_{0,t_3} = 2\right)}$$

$$= \frac{\left(st_3\lambda e^{-st_3\lambda}(r-s)t_3\lambda e^{-(r-s)t_3\lambda}e^{-\lambda t_3(1-r)} + \frac{(\lambda st_3)^2 e^{-\lambda st_3}}{2}e^{-\lambda t_3(1-s)}\right)}{\frac{(\lambda t_3)^2 e^{-\lambda t_3}}{2}}$$

$$= \frac{\left(s(r-s){t_3}^2\lambda^2 e^{-\lambda t_3} + \frac{\lambda^2 s^2 {t_3}^2 e^{-\lambda t_3}}{2}\right)}{\frac{\lambda^2 {t_3}^2 e^{-\lambda t_3}}{2}} = \left(2s(r-s) + s^2\right) = 2sr - s^2$$

Thus joint density function is equal to $f = \frac{\partial^2 \mathrm{F}}{\partial s\, \partial \mathrm{r}} = 2$ is constant. So, $\left(\frac{T_1}{T_3}, \frac{T_2}{T_3}\right)$ has uniform conditional distribution.

Problem 2

Suppose $U_1, \dots, U_n$ are independent random variables each distributed Uniform [0,1]. Suppose [0,1] is partitioned into disjoint intervals $I_1, I_2, \dots, I_m$ with lengths $p_1, \dots, p_m$. Write X_j for the number of U_i's that land in I_j. Find $P(X_1 = k_1, \dots, X_m = k_m)$ for all sets of nonnegative integers $k_1, k_2, \dots, k_m$ that sum to n.

Solution

There are $\frac{n!}{k_1! k_2! \cdot \ldots \cdot k_m!}$ ways to choose which of U_i will be in corresponding intervals, in such way, that in first interval there will be k_1 variables, in second – k_2 variables and so on.

Probability that $U_i \in I_j$ is p_j. Probability that every variable will be in corresponding interval, is equal to $p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$.

So, $P(X_1 = k_1, \dots X_n = k_n) = \frac{n!}{k_1! k_2! \cdot \ldots \cdot k_m!} p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$.

Problem 3

Let $U_{(i)}$ denote the order statistics for a sample of size 6 from the Uniform(0,1) distribution.

(i) Find the joint density function for $U_{(1)}, U_{(2)}, U_{(3)}$.

(ii) Find the joint distribution of $\left(U_{(1)}, U_{(2)}\right)$ conditional on $U_{(3)} = u_3$.

Solution

i)

Denote $\xi_{a,b} = \sum_{k=1}^{k=6} I\left(U_k \in [a,b)\right)$

$$\begin{aligned} &P\left(u_1 \le U_{(1)} < u_1+\delta, u_2 \le U_{(2)} < u_2+\delta, u_3 \le U_{(3)} < u_3+\delta\right) \\ &\qquad = P\big(\xi_{0,u_1} = 0, \xi_{u_1,u_1+\delta} = 1, \xi_{u_1+\delta,u_2} = 0, \xi_{u_2,u_2+\delta} = 1, \xi_{u_2+\delta,u_3} = 0, \xi_{u_3,u_3+\delta} \\ &\qquad = 1, \xi_{u_3+\delta,1} = 3\big) = P^* \end{aligned}$$

Next, let's use problem 2. We have disjoint intervals $(0,u_1),(u_1,u_1+\delta),\dots,(u_3+\delta,1)$ and numbers of points have to be inside it. So,

$$P^* = \frac{6!}{1!\,1!\,1!\,3!}\delta\delta\delta(1-u_3-\delta)^3 = 120\delta^3(1-u_3-\delta)^3$$

Joint density function:

$$\begin{aligned} f_{U_1,U_2,U_3}(u_1,u_2,u_3) &= \lim_{\delta\to 0}\frac{P\left(u_1 \le U_{(1)} < u_1+\delta, u_2 \le U_{(2)} < u_2+\delta, u_3 \le U_{(3)} < u_3+\delta\right)}{\delta^3} \\ &= 120(1-u_3)^3 \end{aligned}$$

Finally,

$$f_{U_1,U_2,U_3}(u_1,u_2,u_3) = \begin{cases} 120(1-u_3)^3, 0 < u_1 < u_2 < u_3 < 1 \\ 0, otherwise \end{cases}$$

ii)

Let's find cumulative distribution function:

$$\begin{aligned} F_{U_{(1)},U_{(2)}}(u_1,u_2) &= P\left(U_{(1)} \le u_1, U_{(2)} \le u_2 \middle| U_{(3)} = u_3\right) \\ &= P\left(\xi_{0,u_1} = 1, \xi_{u_1,u_2} = 1 \middle| U_{(3)} = u_3\right) + P\left(\xi_{0,u_1} = 2, \xi_{u_1,u_2} = 0 \middle| U_{(3)} = u_3\right) \\ &= P\left(\xi_{0,u_1} = 1, \xi_{u_1,u_2} = 1 \middle| \xi_{u_3,1} = 4\right) + P\left(\xi_{0,u_1} = 2, \xi_{u_1,u_2} = 0 \middle| \xi_{u_3,1} = 4\right) \\ &= \frac{P\left(\xi_{0,u_1} = 1, \xi_{u_1,u_2} = 1, \xi_{u_3,1} = 4\right)}{P\left(\xi_{u_3,1} = 4\right)} + \frac{P\left(\xi_{0,u_1} = 2, \xi_{u_1,u_2} = 0,\ \xi_{u_3,1} = 4\right)}{P\left(\xi_{u_3,1} = 4\right)} \\ &= \frac{\left(\frac{6!}{4!}u_1(u_2-u_1)(1-u_3)^4 + \frac{6!}{2!\,4!}u_1^2(1-u_3)^4\right)}{\frac{6!}{2!\,4!}u_3^2(1-u_3)^4} = \frac{30u_1(u_2-u_1)+15u_1^2}{15u_3^2} \\ &= \frac{u_1(2u_2-u_1)}{u_3^2} \end{aligned}$$

So, $F_{U_{(1)},U_{(2)}}(u_1,u_2) = \begin{cases} \frac{u_1(2u_2-u_1)}{u_3^2}, 0 < u_1 < u_2 < u_3 \\ 0, otherwise \end{cases}$

Problem 4

Suppose X and Y are random variables and α and β are positive constants such that X has a Poisson(βy) distribution conditional on $Y = y$ and $Y \sim$gamma(α). Find the marginal distribution for X.

X is discrete random variable, so we should find marginal distribution as $P(X = x)$ for all whole $x \geq 0$

$$P(X = x) = \int_{y=-\infty}^{y=\infty} P(X = x|Y = y) f_Y(y) dy = \int_0^{\infty} \frac{(\beta y)^x e^{-\beta y}}{x!} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy$$

$$= \frac{\beta^x}{x!\,\Gamma(\alpha)} \int_0^{\infty} y^{x+\alpha-1} e^{-y(1+\beta)} dy$$

$$= \frac{\beta^x}{(1+\beta)^{x+\alpha} x!\,\Gamma(\alpha)} \int_0^{\infty} \big(y(1+\beta)\big)^{x+\alpha-1} e^{-y(1+\beta)} d\big((1+\beta)y\big) = \frac{\beta^x \Gamma(\mathrm{x}+\mathrm{\alpha})}{(1+\beta)^{x+\alpha} x!\,\Gamma(\alpha)}$$