**Problem**

a) Find $b \in \mathbb{Z}$ such that $\left(\frac{b}{p}\right) = -1$ for all odd primes $p \equiv 2\ (mod\ 3)$

b) Find $b \in \mathbb{Z}$ such that $\left(\frac{b}{p}\right) = -1$ for all odd primes $p \equiv 2\ (mod\ 5)$

c) Let q be an odd prime, and let $a \in \mathbb{Z}$ such that $\left(\frac{a}{q}\right) = -1$. Show that there exists $b \in \mathbb{Z}$ such that $\left(\frac{b}{p}\right) = -1$ for all odd primes $p \equiv a\ (mod\ q)$

d) Now suppose instead that $\left(\frac{a}{q}\right) = 1$. Show that there does not exists $b \in \mathbb{Z}$ such that $\left(\frac{b}{p}\right) = -1$ for all odd primes $p \equiv a\ (mod\ q)$

**Solution**

a)

I will prove that -3 is quadratic non-residue modulo p, if p is congruent to 2 modulo 3.

To do this, I will use Lagrange symbol and its properties, such as quadratic reciprocity law:

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$$

So,

$$\left(\frac{-3}{p}\right) = \left(\frac{-1 * 3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{3}{p}\right) = (using\ reciprocity\ law)$$
$$= (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{3-1}{2}\right)} = (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}} = \left(\frac{p}{3}\right)$$
$$= (p\ is\ congruent\ to\ 2\ modulo\ 3) = \left(\frac{2}{3}\right) = -1$$

So, we just proved required statement for $b = -3$.

b)

I will in the same way prove that 5 is quadratic non-residue modulo p, if p is congruent to 2 modulo 5.

$$\left(\frac{5}{p}\right) = (reciprocity) = \left(\frac{p}{5}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{5-1}{2}\right)} = \left(\frac{p}{5}\right)(-1)^{p-1} = (p\ is\ odd) = \left(\frac{p}{5}\right)$$
$$= \left(\frac{2}{5}\right) = -1$$

So, the statement is proved.

c)

Lets look at two cases.

First . $\frac{q-1}{2}\ is\ even$. Then number b=(q) is non-residue modulo p. Proof:

$$\left(\frac{q}{p}\right) = (reciprocity) = \left(\frac{p}{q}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} = \left(\frac{a}{p}\right) = -1$$

Second. $\frac{q-1}{2}\ is\ odd$. b=-q is required number. Proof:

$$\left(\frac{-q}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{q}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$$
$$= (-1)^{\frac{p-1}{2}}\left(\frac{p}{q}\right)(-1)^{\left(\frac{p-1}{2}\right)} = \left(\frac{p}{q}\right) = \left(\frac{a}{q}\right) = -1$$

So, we found b for all odd primes congruent to a modulo q, where $\left(\frac{a}{q}\right) = -1$, such that $\left(\frac{b}{p}\right) = -1$

d)

Suppose there exist such number b.

Then, using quadric reciprocity law we have:

$$1 = \left(\frac{b}{p}\right) = \left(\frac{p}{b}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{b-1}{2}\right)}$$

Again we have two cases.

First : $\frac{b-1}{2}\ is\ even.$

Then we have that $\left(\frac{p}{b}\right) = 1$

But, using Dirichlet's theorem, we may choose prime p, such that: $(p \equiv a\ mod\ q)\ and\ (p \equiv b_0\ mod\ b), where\ b_0 is\ any\ non-residue\ modulo\ b.$

For such prime $\left(\frac{p}{b}\right) = \left(\frac{b_0}{b}\right) = -1$.

Contradiction.

Second case.

$$\frac{b-1}{2}\ is\ odd$$

$$\left(\frac{p}{b}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{b-1}{2}\right)} = \left(\frac{p}{b}\right)(-1)^{\left(\frac{p-1}{2}\right)}$$

But again, using Dirichlet 's theorem we may choose p, such that $p \equiv 1\ mod\ 4\ \left(then \frac{p-1}{2}\ is\ even\right), p \equiv a\ mod\ q\ and\ p \equiv b_0 mod\ b$

Then $\left(\frac{p}{b}\right)(-1)^{\left(\frac{p-1}{2}\right)} = \left(\frac{p}{b}\right) = \left(\frac{b_0}{b}\right) = -1$

We have contradiction again.

In proofs I used multiple conditions on p, while choosing it by Dirichlet's theorem. But they may be done as one, firstly used china remainder theorem.

So, the proof is finished.