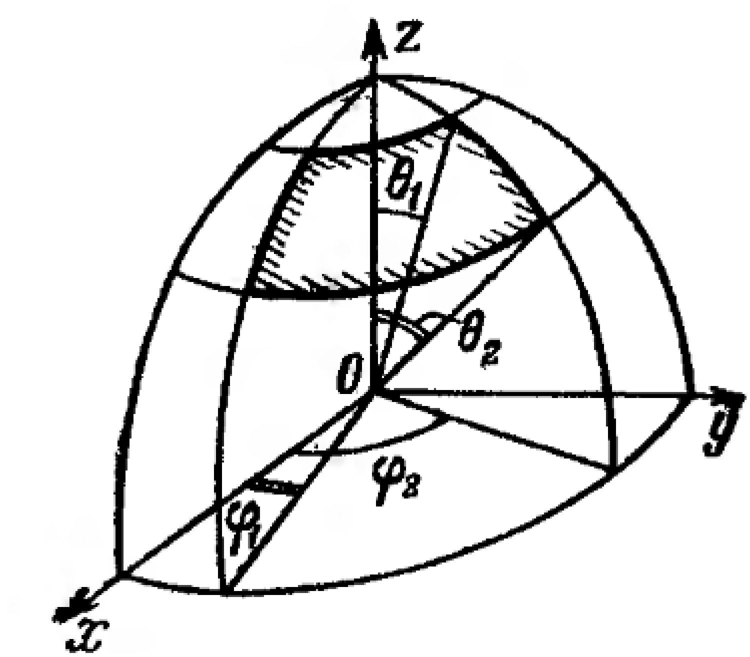
Problem 1

Find the area of the part of the sphere between two meridians $\phi_1 < \phi < \phi_2$ and two parallels $\theta_1 < \theta < \theta_2$.



Solution

In spherical coordinates $x = R\sin\theta\cos\phi\,, y = R\sin\theta\sin\phi\,, z = R\cos\theta$

I will use formulas:

$$S = \int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\sqrt{EG - F^2}\,d\phi d\theta$$

where

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$F = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \theta}$$

$$G = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

Matrix of derivatives:

$$\begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} -R\sin\theta\sin\phi & R\sin\theta\sin\phi & 0 \\ R\cos\theta\cos\phi & R\cos\theta\sin\phi & -R\sin\theta \end{pmatrix}$$

$$E = R^2\sin^2\theta$$

$$F = 0$$

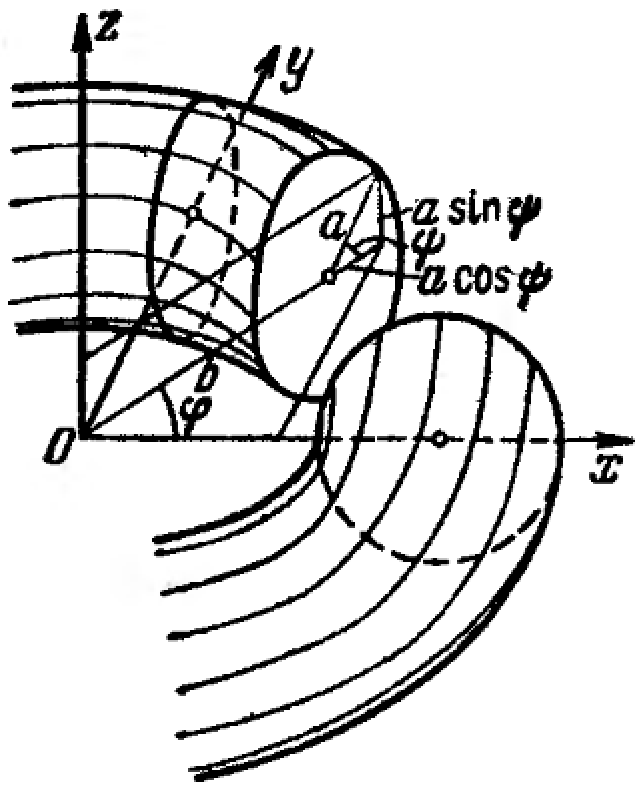
$$G = R^2$$

So,

$$S = R^2\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\sin\theta\, d\phi d\theta = R^2(\phi_1 - \phi_2)(\cos\theta_1 - \cos\theta_2)$$

Problem 2

Find the area of the torus received by circle rotation around axis z. The circle initially was located in x-z plane with center at point (b,0,0) and radius a (b>a).



Solution

Parameterization of the torus:

$$\begin{cases} x = (b + a\cos\psi)\cos\phi \\ y = (b + a\cos\psi)\sin\phi \\ z = a\sin\psi \end{cases}$$

Matrix of derivatives:

$$\begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} -(b + a\cos\psi)\sin\phi & (b + a\cos\psi)\cos\phi & 0 \\ -a\sin\psi\cos\phi & -a\sin\psi\sin\phi & a\cos\psi \end{pmatrix}$$

Now we may calculate the area as the integral:

$$S = \int_0^{2\pi}\int_0^{2\pi} \sqrt{EG - F^2}\, d\phi d\psi = a\int_0^{2\pi}\int_0^{2\pi} (b + a\cos\psi)\, d\phi d\psi = 4\pi^2 ab$$