Problem

The closed cylindrical container was received after connecting two cylindrical containers each filled by homogeneous gas, properties and concentration of environment are different. Work out the boundary problem about diffusion in composite container, considering cases:

- cylinders are connected directly
- cylinders are connected via semipermeable membrane.

Solution

Suppose cylinders are located along axis x, $-L_1 < x < 0, 0 < x < L_2$, have identical area of section S and are connected at point 0. Let c_i, D_i are coefficients of porosity and diffusion. Let U(x,t) is concentration at t>0. For arbitrary point except x=0 we may write usual equation of diffusion:

$$c_1 \frac{\partial U(x,t)}{\partial t} = D_1 \frac{\partial^2 U(x,t)}{\partial x^2}, \qquad -L_1 < x < 0, t > 0$$

$$c_2 \frac{\partial U(x,t)}{\partial t} = D_2 \frac{\partial^2 U(x,t)}{\partial x^2}, \qquad 0 < x < L_2, t > 0$$

Consider point x=0. If cylinders are connected straight, concentration is continuous:

$$U(0+0,t) = U(0-0,t)$$

For elementary volume $[-dx, dx]$ law of conservation the mass is

$$\left(c_1\big(U(\xi_1, t+dt) - U(\xi_1,t)\big) + c_2\big(U(\xi_2, t+dt) - U(\xi_2,t)\big)\right) dxS$$
$$= \left(-D_1 \frac{\partial U(-dx,t)}{\partial x} + D_2 \frac{\partial U(dx,t)}{\partial x}\right) dtS$$

After boundary transition we have:

$$D_1 \frac{\partial U(0-0,t)}{\partial x} = D_2 \frac{\partial U(0+0,t)}{\partial x}$$

If cylinders are connected via semipermeable membrane, consider elementary volume [-dx,0] and let's work out the equation of mass balance.

$$dQ_1 = c_1\big(U(\xi_1, t+dt) - U(\xi_1,t)\big) S dx$$

is mass of gas that is used for changing concentration.

$$dQ_2 = -D_1 \frac{\partial U(-dx,t)}{\partial x} S dt$$

is a flow of gas through $x = -dx$ for time dt.

$$dQ_3 = \alpha\big(U(0-0,t) - U(0+0,t)\big)$$

is a flow of gas through x=0

Balance equation is:

$$dQ_1 = dQ_2 + dQ_3$$

or

$$c_1\big(U(\xi_1, t+dt) - U(\xi_1,t)\big)Sdx = -D_1 \frac{\partial U(-dx,t)}{\partial x} Sdt + \alpha\big(U(0-0,t) - U(0+0,t)\big)$$

After boundary transition:

$$0 = -D_1 \frac{\partial U(0-0,t)}{\partial x} + \alpha\big(U(0-0,t) - U(0+0,t)\big)$$

For [0;dx] we have:

$$0 = -D_2 \frac{\partial U(0+0,t)}{\partial x} + \alpha\big(U(0+0,t) - U(0-0,t)\big)$$

Initial conditions:

$$U(x,0) = U_1, -L_1 < x < 0; \quad U(x,0) = U_2, 0 < x < L_2$$

Boundary conditions:

$$\frac{\partial U(-L_1,t)}{\partial x} = \frac{\partial U(L_2,t)}{\partial x}, t > 0$$