Problem

Prove following functions are primitive recursions:

a) $k^n(x_1, x_2, \dots, x_n) = k$

b) $f(x_1, x_2) = x_1 + x_2$

c) $f(x_1, x_2) = x_1 x_2$

d) $f(x_1, x_2) = x_1^{x_2}$

e) $sg(x_1) = \begin{cases} 0, x_1 = 0 \\ 1, x_1 \geq 1 \end{cases}$

f) $nsg(x_1) = \begin{cases} 1, x_1 = 0 \\ 0, x_1 \geq 1 \end{cases}$

g) $f(x_1, x_2) = \begin{cases} x_1 - x_2, x_1 \geq x_2 \\ \quad 0, \ x_1 < x_2 \end{cases}$

Solution

a)

Zero-function is $S^2(0, I_1^n)$, constant k is $S^2\big(s, S^2(s, \dots S^2(0, I_1^n) \dots)\big)$

b)

$$f(x_1, 0) = x_1 = I_1^1(x_1)$$

$$f(x_1, x_2 + 1) = x_1 + (x_2 + 1) = (x_1 + x_2) + 1 = s(x_1 + x_2) = s\big(f(x_1, x_2)\big)$$

So, f is primitive recursion of $g(x_1) = I_1^1(x_1)$ and $h(x_1, x_2, x_3) = x_3 + 1 = s(x_3) = S^2(s, I_3^3)(x_1, x_2, x_3)$

c)

$$f(x_1, 0) = 0 = o(x_1)$$

$$f(x_1, x_2 + 1) = x_1(x_2 + 1) = x_1 x_2 + x_1 = f(x_1, x_2) + x_1$$

So, f is primitive recursive of $g(x_1) = o(x_1)$ and $h(x_1, x_2, x_3) = x_3 + x_1$. Using b) h is primitive recursion, so f primitive recursion too.

d)

$$f(x_1, 0) = x_1^0 = 1$$

$$f(x_1, x_2 + 1) = x_1^{x_2+1} = x_1^{x_2} x_1 = f(x_1, x_2) x_1$$

Using c) multiplication is primitive, so f is primitive too.

e)

$$sg(0) = 0 = o(x_1)$$

$$sg(x_1 + 1) = 1$$

f)

$$nsg(0) = 1 = s\big(o(x_1)\big)$$

$$nsg(x_1 + 1) = 0$$

g)

At fist let’s show $f(x_1, 1)$ is primitive. Really,

$$f(0,1) = o(x_1)$$

$$f(x_1 + 1,1) = x_1 = I_1^2(x_1, x_2)$$

Now let’s show f is primitive.

$$f(x_1, 0) = x_1 = I_1^1(x_1)$$

$$f(x_1, x_2 + 1) = f(f(x_1, x_2), 1)$$

The proof is finished.