Problem

Consider a competitive Lotka-Volterra equations:

$$\begin{cases} x_1' = x_1(3 - x_2 - x_1) \\ x_2' = x_2(5 - 2x_1 - x_2) \end{cases}$$

Find stationary points, coefficients of linearized system, determine type of each stationary point, build a phase portrait.

Solution.

Firstly we solve system of equations:

$$\begin{cases} x_1(3 - x_2 - x_1) = 0 \\ x_2(5 - 2x_1 - x_2) = 0 \end{cases}$$

We got 4 stationary states:

I) $x_1^I = 0, x_2^I = 0$

II) $x_1^{II} = 0, x_2^{II} = 5$

III) $x_1^{III} = 3, x_2^{III} = 0$

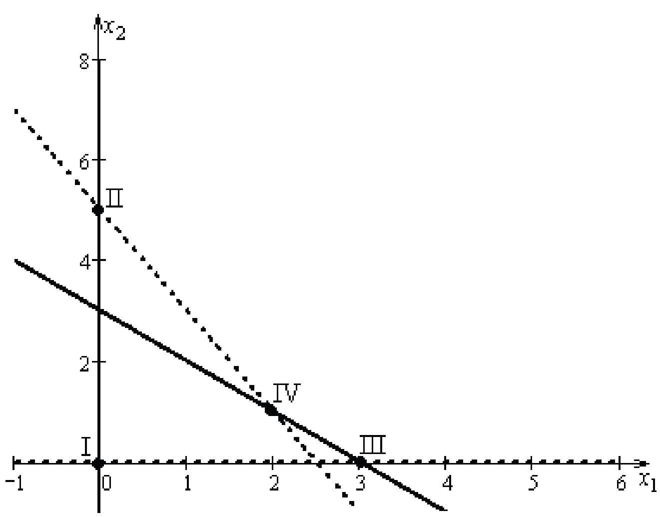
IV) $x_1^{IV} = 2, x_2^{IV} = 1$

Next we will build main isoclinic lines.

Isoclinic line of horizontal tangent lines: $x_2 = 0, x_2 = 5 - 2x_1$

Isoclinic line of vertical tangent lines: $x_1 = 0, x_2 = 3 - x_1$

Plotting this:



Linearization of the system in the neighborhood of the stationary state:

$$P'_{x_1}(x_1,x_2) = 3 - x_2 - 2x_1, P'_{x_2}(x_1,x_2) = -x_1$$

$$Q'_{x_1}(x_1,x_2) = -2x_2, Q'_{x_2}(x_1,x_2) = 5 - 2x_1 - 2x_2$$

In the neighborhood of $x_1^I = 0, x_2^I = 0$ matrix of coefficients of linearized system is $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$. Roots of the corresponding characteristic equation are $\lambda_{1,2}^I = \left[\begin{matrix} 3 \\ 5 \end{matrix}\right.$. Roots are real and positive. So stationary state $x_1^I = 0, x_2^I = 0$ is unsteady.

In the neighborhood of $x_1^{II} = 0, x_2^{II} = 5$ matrix of coefficients of linearized system is $\begin{pmatrix} -2 & 0 \\ -10 & -5 \end{pmatrix}$. Roots of the corresponding characteristic equation are $\lambda_{1,2}^{II} = \left[\begin{matrix} -2 \\ -5 \end{matrix}\right.$. Roots are real and negative. So stationary state $x_1^{II} = 0, x_2^{II} = 5$ is steady state.

In the neighborhood of $x_1^{III} = 3, x_2^{III} = 0$ matrix of coefficients of linearized system is $\begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix}$. Roots of the corresponding characteristic equation are $\lambda_{1,2}^{III} = \left[\begin{matrix} -3 \\ -1 \end{matrix}\right.$. Roots are real and negative. So stationary state $x_1^{III} = 3, x_2^{III} = 0$ is steady state.

In the neighborhood of $x_1^{IV} = 2, x_2^{IV} = 1$ matrix of coefficients of linearized system is $\begin{pmatrix} -2 & -2 \\ -2 & -1 \end{pmatrix}$. Determinant of the matrix is -2<0. So stationary state $x_1^{IV} = 2, x_2^{IV} = 1$ is unsteady state.

The phase portrait is: