Problem 1

Find

$$A = \lim_{R\to\infty} \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta$$

Solution

Using inequality $\sin\theta \geq \frac{2}{\pi}\theta$ at interval $0 \leq \theta \leq \frac{\pi}{2}$ we have $e^{-R\sin\theta} \leq e^{-\frac{2}{\pi}\theta}$. So,

$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \leq \int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}R\theta}\, d\theta = \frac{\pi}{2R}\left(1 - e^{-R}\right)$$

So we have:

$$0 \leq A \leq \lim_{R\to\infty} \frac{\pi}{2R}\left(1 - e^{-R}\right) = 0$$

that implies $A = 0$.

Problem 2

Let f is continuous at [A;B]. Prove that

$$\lim_{h\to 0}\frac{1}{h}\int_a^x \big(f(t+h)-f(t)\big)dt = f(x)-f(a),\ \ A<a<x<B$$

Solution

Consider the antiderivative function F of function f. Then we have:

$$\int_a^x \big(f(t+h)-f(t)\big)dt = \int_a^x \big(F'(t+h)-F'(t)\big)dt = F(t+h)-F(t)|_{t=a}^{t=x}$$
$$= F(x+h)-F(x)-\big(F(a+h)-F(a)\big)$$

So,

$$\lim_{h\to 0}\frac{1}{h}\int_a^x \big(f(t+h)-f(t)\big)dt = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}-\frac{F(a+h)-F(a)}{h}$$
$$= F'(x)-F'(a) = f(x)-f(a)$$

Problem 3

Find $F'(a)$ if:

$$F(a) = \int_0^a f(x+a, x-a)dx$$

Function f is continuously differentiable.

Solution

Using continuity of partial derivatives of $f(u,v)$ and Leibniz formula we have:

$$F'(a) = f(2a,0) + \int_0^a \left(f_u'(u,v) - f_v'(u,v)\right)dx$$

Using

$$\frac{df}{dx} = f_u' + f_v'$$

we have:

$$\int_0^a \left(f_u'(u,v) - f_v'(u,v)\right)dx = 2\int_0^a f_u' dx - f(2a,0) + f(a,-a)$$

So,

$$F'(a) = f(a,-a) + 2\int_0^a f_u' dx$$

Problem 4

Find $F''(x)$ if

$$F(x) = \frac{1}{h^2}\int_0^h d\xi \int_0^h f(x+\xi+\mu)d\mu$$

$h > 0$ and f is continuous.

Solution

Using formula

$$\int_\alpha^\beta f(t+w)dt = \int_{\alpha+w}^{\beta+w} f(t)dt$$

and differentiating on parameter we have:

$$F'(x) = \frac{d}{dx}\left(\frac{1}{h^2}\int_0^h d\xi \int_{x+\xi}^{h+x+\xi} f(\mu)d\mu\right) = \frac{1}{h^2}\int_0^h \big(f(h+x+\xi) - f(x+\xi)\big)d\xi$$
$$= \frac{1}{h^2}\left(\int_{x+h}^{x+2h} f(\xi)d\xi - \int_x^{x+h} f(\xi)d\xi\right)$$

Differentiating again we get:

$$F''(x) = \frac{1}{h^2}\big(f(2h+x) - 2f(h+x) + f(x)\big)$$