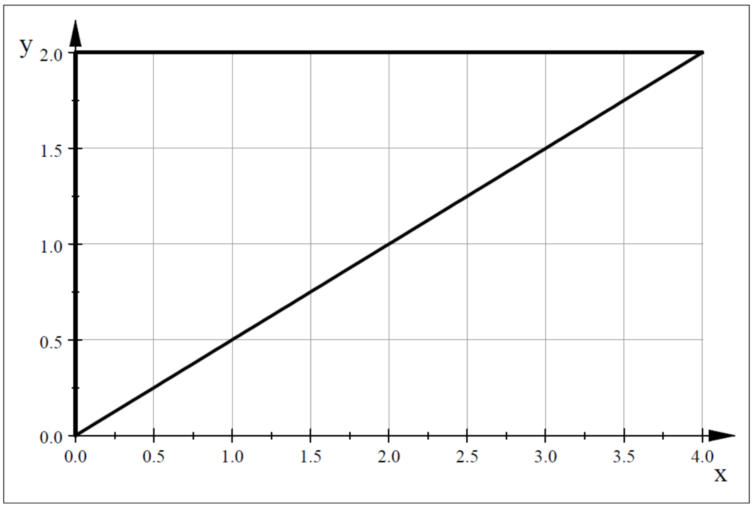
Problem 1

Evaluate

$$\iint_R e^{y^2} dA$$

where R is a region shown below.



Solution.

To integrate this function over R firstly I will transform this integral into two nested integrals.

To do this I will describe R in terms of inequalities.

$$0 \le y \le 2$$

$$0 \le x \le 2y$$

So,

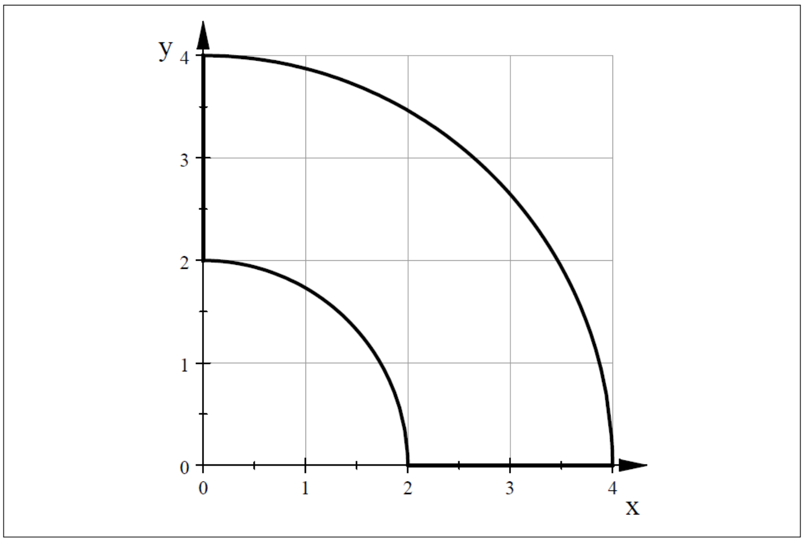
$$\iint_R e^{y^2} dA = \int_0^2 \int_0^{2y} e^{y^2} dxdy = \int_0^2 2ye^{y^2} dy = \int_0^2 e^{y^2} dy^2 = e^{y^2}|_{y=0}^{y=2} = e^4 - 1$$

Problem 2

Evaluate

$$\iint_R (x+y)dA$$

where R is a quarter annulus shown below. The arcs are parts of circles.



Solution.

To evaluate this integral I will use polar system of coordinates.

$$\iint_R (x+y)dA = \left| polar\ system: \begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}\ determinant\ of\ Jacobian\ matrix\ D \right.$$

$$\left. = r; region\ in\ polar\ coordinates: 2 \le R \le 4, 0 \le \phi \le \frac{\pi}{2} \right|$$

$$= \int_2^4 \int_0^{\frac{\pi}{2}} r(\cos\phi + \sin\phi) r\, d\phi\, dr$$

$$= (inner\ integral\ is\ taking\ on\ \phi, so\ we\ can\ remove\ r^2\ from\ the\ inner\ integral)$$

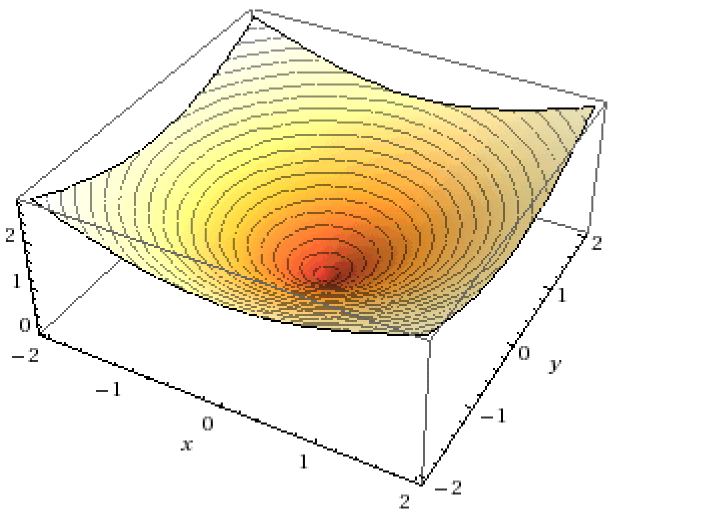
$$= \int_2^4 r^2 \int_0^{\frac{\pi}{2}} (\cos\phi + \sin\phi)\, d\phi\, dr = \int_2^4 2r^2 dr = \frac{2r^3}{3} \Big|_{r=2}^{r=4} = \frac{2}{3}(4^3 - 2^3) = 2 * \frac{56}{3} = \frac{112}{3}$$

Problem 3

Give an integral in rectangular coordinates for the surface area of the portion of the cone $z^2 = x^2 + y^2$ that lies above the circular region $x^2 + y^2 \leq 4$ in the x,y-plane and then find the surface area. Sketch the surface area to be found.

Solution

At first lets sketch area that we will integrate:



Explicit expression of z:

$$z = \sqrt{x^2 + y^2}$$

Surface area may be found using formula:

$$S = \iint_R \sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1dxdy}$$

Some pre-calculation:

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} \quad \frac{dz}{dy} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$S = \int_{x^2+y^2 \leq 4} \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1\,dxdy} = \int_{x^2+y^2 \leq 4} \sqrt{2}dxdy = \sqrt{2} \int_{x^2+y^2 \leq 4} dxdy$$
$$= \sqrt{2}S(x^2 + y^2 \leq 4) = \sqrt{2}\pi 2^2 = 4\sqrt{2}\pi$$

Here I used that $\iint_R dxdy = S(R)$ - area of R, and used formula for area of the circle.

Problem 4

Give a triple integral in **cylindrical** coordinates for the volume that lies both within the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$. DO NOT EVALUATE THIS INTEGRAL.

Solution.

At first I will find point of intersections of cylinder and sphere.

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 + z^2 = 4 \end{cases}$$

Solution is $x^2 + y^2 = 1, z = \pm\sqrt{3}$

Now let's determine borders for coordinates.

Cylindrical system of coordinates:

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = h \end{cases}$$

Firstly h changes from -2 to 2.

Next, if $h \in [-2, -\sqrt{3}], then\ \ r \in [0, \sqrt{4 - h^2}], \phi \in [0, 2\pi]$

If $h \in [-\sqrt{3}, \sqrt{3}]$, $then\ r \in [0,1], \phi \in [0, 2\pi]$

If $h \in [\sqrt{3}, 2], then\ \ r \in [0, \sqrt{4 - h^2}], \phi \in [0, 2\pi]$

Determinant of the Jacobian is equal to r.

Finally,

$$V = \int_{-2}^{-\sqrt{3}} \int_{0}^{\sqrt{4-h^2}} \int_{0}^{2\pi} r\ d\phi\ dr\ dh + \int_{-\sqrt{3}}^{\sqrt{3}} \int_{0}^{1} \int_{0}^{2\pi} r\ d\phi\ dr\ dh + \int_{\sqrt{3}}^{2} \int_{0}^{\sqrt{4-h^2}} \int_{0}^{2\pi} r\ d\phi\ dr\ dh$$

Problem 5

Evaluate

$$\iiint_V e^{\sqrt{x^2+y^2+z^2}}\,dV$$

where V s the part of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

Solution.

To evaluate this I will use spherical coordinates.

$$\begin{cases} x = r\cos\theta\cos\phi \\ y = r\cos\theta\sin\phi \\ z = r\sin\theta \end{cases}$$

Jacobian determinant: D=$r^2\cos\theta$

New borders of integration:

$$0 \le r \le 3$$

$$0 \le \phi \le \frac{\pi}{2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$\iiint_V e^{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^3\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} e^r r^2\cos\theta\;d\theta\,d\phi dr = \int_0^3\int_0^{\frac{\pi}{2}} e^r r^2 d\phi dr = \frac{\pi}{2}\int_0^3 e^r r^2 dr$$
$$= \frac{\pi}{2}e^r(r^2 - 2r + 2)\,|_{r=0}^{r=3} = \frac{\pi}{2}(5e^3 - 2)$$