Problem 1

Every pupil of the first class is friend to 3 pupils from the second class. Every pupil of the second class is friend to 3 pupils from the first class. Prove number of pupils in these classes is equal.

Solution

Consider an undirected graph G which vertices are pupils. There is an edge between two vertices in the graph if and only if corresponding pupils are friends. Since we take into consideration only friendship between pupils of different classes then graph G is bipartite. Let the first class corresponds to the part A of the graph containing m vertices, and the second class corresponds to the part B of the graph, containing n vertices. Number of edges connecting part A and part B equals 3m (because every pupil from part A has 3 friends in part B). On the other hand number of edges equals 3n (because every pupil from part B has 3 friends in part A). Since these edges are from the one graph $3n = 3m$ that implies $n = m$. So number of pupils in these classes is equal.

Problem 2

Every pupil from the first class is friend to at least half of pupils of the second class, and every pupil from the second class is friend to at most half of pupils of the first class. Prove every pupil from the first class is friend to exactly half of pupils of the second class and every pupil from the second class is friend to exactly half of pupils of the first class.

Solution

Let's build the bipartite graph as in problem 1. The first class corresponds to part A with m vertices and the second class corresponds to the part B with n vertices.

Every vertex u from the part A has degree $d(u) \geq \frac{n}{2}$. Every vertex v from the part B has degree $d(v) \leq \frac{m}{2}$. For number of edges coming out from part A holds

$$|E_1| \geq m \cdot \frac{n}{2}$$

For number of edges coming out from part B holds

$$|E_2| \leq n \cdot \frac{m}{2}$$

Since edges coming out from A and coming out from B form the same set then $|E_1| = |E_2|$. This equality holds if and only if every vertex from the part A has degree $\frac{n}{2}$ and every vertex from the part B has degree $\frac{m}{2}$. So every pupil is friend to exactly half of pupils from the other class.