Problem 1

Prove Heron's formula for the area of the triangle

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

where a,b,c are lengths of the triangle sides and $s = \frac{a+b+c}{2}$ is the semiperimeter of the triangle.

Solution

Area of the triangle equals

$$S = \frac{1}{2}ab\sin\gamma$$

where γ is the angle opposite to c. Using law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

This implies

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

So

$$\begin{aligned}\sin^2\gamma = 1-\cos^2\gamma &= (1-\cos\gamma)(1+\cos\gamma)\\ &= \frac{2ab-a^2-b^2+c^2}{2ab}\cdot\frac{2ab+a^2+b^2-c^2}{2ab}\\ &= \frac{c^2-(a-b)^2}{2ab}\cdot\frac{(a+b)^2-c^2}{2ab}\\ &= \frac{1}{4a^2b^2}(c-a+b)(c+a-b)(a+b-c)(a+b+c)\end{aligned}$$

Using $s = \frac{a+b+c}{2}$ this formula may be rewritten as

$$\sin^2\gamma = \frac{4}{a^2b^2}\big(s(s-a)(s-b)(s-c)\big)$$

Since $0 < \gamma < \pi$ $\sin\gamma > 0$. So

$$\sin\gamma = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

Substituting this to the area formula we get

$$S = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ab\ \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(s-c)}$$

The statement is proved.

Problem 2

Prove formulas

$$R = \frac{abc}{4S};\quad r = \frac{2S}{a+b+c}$$

where a,b,c are lengths of the triangle sides, S is area of the triangle and r,R are radiuses of the incircle and the circumcircle.

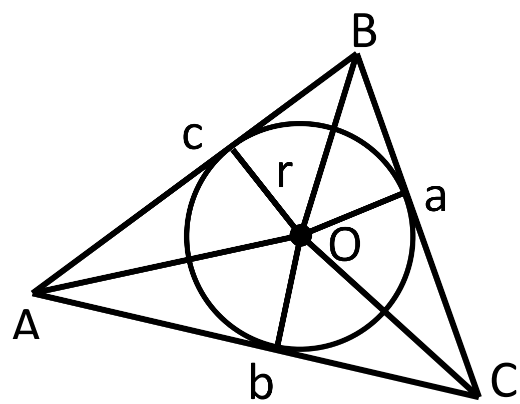
Solution

As we know

$$R = \frac{a}{2\sin\alpha}$$

where α is the angle opposite to side a of the triangle. Multiplying numerator and denominator of the fraction by bc and using $S = \frac{1}{2}bc\sin\alpha$:

$$R = \frac{abc}{4S}$$



Area of the triangle ABC equals sum of areas of the triangles OAB, OBC and OCA.

$$S_{ABC} = S_{OAB} + S_{OBC} + S_{OCA}$$

$$S_{OAB} = \frac{1}{2}cr$$

$$S_{OBC} = \frac{1}{2}ar$$

$$S_{OCA} = \frac{1}{2}br$$

So

$$S = \frac{1}{2}r(a + b + c)$$

$$r = \frac{2S}{a + b + c}$$

The formulas are proved.