Problem 1

Let H is a Hilbert space, $A \in \mathcal{L}(H)$. Prove that:

1) $A^{**} = (A^*)^* = A$

2) $\|A^*\| = \|A\|$

3) If $A = A^*$ then $\|A^2\| = \|A\|^2$

Solution.

1)

For $x, y \in H$ holds:

$$(A^*x, y) = \overline{(y, A^*x)} = \overline{(Ay, x)} = (x, Ay)$$

Since $(A^*x, y) = (x, A^{**}y)$ then we have $(x, A^{**}y) = (x, Ay)$.

Let $x = A^{**}y - Ay$. Then $\|A^{**}y - Ay\|^2 = 0$.

So, $A^{**}y = Ay$ for each y, that means $A = A^{**}$.

2)

We have:

$$|(x, A^*y)| = |(Ax, y)| \le \|Ax\|\|y\| \le \|A\|\|x\|\|y\|$$

Taking $x = A^*y$ we got:

$$\|A^*y\|^2 \le \|A\|\|A^*y\|\|y\|$$

that follows $\|A^*y\| \le \|A\|\|y\|$, so $\|A^*\| \le \|A\|$.

To prove inequality to other side let's use part 1:

$$\|A\| = \|A^{**}\| \le \|A^*\|$$

The statement is proved.

3)

Using $\|A^2x\| \le \|A\|\|Ax\| \le \|A\|^2\|x\|$, so $\|A^2\| \le \|A\|^2$

Let's prove that $\|A^2\| \ge \|A\|^2$. Really,

$$\|Ax\|^2 = (Ax, Ax) = (A^2x, x) \le \|A^2x\|\|x\| \le \|A^2\|\|x\|^2$$

that implies

$$\|Ax\| \le \sqrt{\|A^2\|}\|x\|$$

so $\|A\| \le \sqrt{\|A^2\|}, or\ \|A\|^2 \le \|A^2\|$. The statement is proved.

Problem 2

Let H is complex separable Hilbert space, A is a operator set by matrix $\left(a_{jk}\right)_{j,k=1}^{\infty}$. Prove that operator A^* is set by matrix $\left(a_{jk}^*\right)_{j,k=1}^{\infty}$, where $a_{jk}^* = \overline{a_{kj}}$.

Solution.

Conjugated operator A^* is set by some matrix $\left(A_{jk}^*\right)_{j,k=1}^{\infty}$, also holds $a_{jk}^* = \left(A^* e_k, e_j\right), j, k \in N$. But $\left(A^* e_k, e_j\right) = \left(e_k, Ae_j\right) = \overline{(Ae_j, e_k)} = \overline{a_{kj}}$, so $a_{jk}^* = \overline{a_{kj}}$.

Problem 3

Let (T, μ) is a space with $\sigma - finite$ measure, A is integral operator in complex Hilbert space $L_2(T, \mu)$ with kernel $K \in L_2(T \times T, \mu \times \mu)$, then A^* is integral operator with kernel $K^* = \overline{K}$.

Solution.

For every $x, y \in L_2(T, \mu)$ we need to find such element y^* that $(Ax, y) = (x, y^*)$. We have:

$$(Ax, y) = \int_T (Ax)(t)\overline{y(t)}\, dt = \int_T \left(\int_T K(t,s)x(s)ds\right)\overline{y(t)}dt = \int_T \int_T K(t,s)x(s)\overline{y(t)}dt\, ds$$
$$= \int_T x(s)\overline{\left(\int_T \overline{K(t,s)}y(t)dt\right)}\, ds$$

Higher we changed order of integration using Fubini's theorem. The equality we get implies

$$(A^* y)(s) = y^*(s) = \int_T \overline{K(t,s)}y(t)dt$$

Renaming variables (t to s and s to t) we found that A^* is an integral operator with kernel $K^* = \overline{K}$.