Problem 1

Prove that equality

$$(A \times B) \cup (B \times A) = K \times K$$

implies $A = B = K$.

Solution

For arbitrary $x \in A; y \in B$

$$(x, y) \in (A \times B) \Rightarrow (x, y) \in K \times K \Rightarrow x \in K, y \in K \Rightarrow A \subseteq K; B \subseteq K$$

From the other side,

$$x \in K \Rightarrow (x, x) \in K \times K \Rightarrow \{using\ equality\} \Rightarrow (x, x) \in (A \times B)\ or\ (x, x) \in (B \times A) \Rightarrow \{(x \in A, x \in B\ or\ x \in B, x \in A) = (x \in A, x \in B)\} \Rightarrow x \in A, x \in B \Rightarrow K \subseteq A; K \subseteq B$$

So $A = B = K$.

Problem 2

For the binary relation

$$\rho = \{(x, y): x^2 = y\}$$

find the domain, range and draw the Cartesian diagram.

Solution

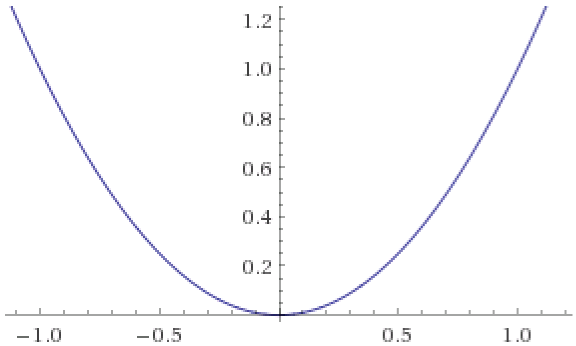
Domain:

$$D_\rho = \{x \in R: \exists y\ (x, y) \in \rho\} = R$$

Range:

$$R_\rho = \{y \in Y: \exists x\ (x, y) \in \rho\} = R_+ \cup 0$$

Cartesian diagram for this relation is



Problem 3

Find inverse relations for

$$\rho_1 = \{(x,y): x = y^2\};\ \ \rho_2 = \{(x,y): x + y \leq 2\};\ \ \rho_3 = \{(x,y): x + y \in Z\}$$

Solution

$$\rho_1^{-1} = \{(x,y): (y,x) \in \rho_1\} = \{(x,y): y = x^2\}$$

$$\rho_2^{-1} = \{(x,y): (y,x) \in \rho_2\} = \{(x,y): y + x \leq 2\} = \rho_2$$

$$\rho_3^{-1} = \{(x,y): (y,x) \in \rho_3\} = \{(x,y): y + x \in Z\} = \rho_3$$

Problem 4

Let X is an arbitrary set. Denote by I_X relation on X

$$I_X = \{(x,x): x \in X\}$$

Prove

$$I_B \circ \rho = \rho; \qquad \rho \circ I_A = \rho$$

Solution

$$\begin{aligned} I_b \circ \rho &= \{(x,y) \in A \times B: \exists z \in B: (x,z) \in \rho, (z,y) \in I_B\} \\ &= \{(x,y) \in A \times B: \exists z \in B: (x,z) \in \rho,, z = y\} \\ &= \{(x,y) \in A \times B: (x,y) \in \rho\} = \rho \end{aligned}$$

$$\rho \circ I_A = \{(x,y) \in A \times B: \exists z \in A: (x,z) \in I_A, (z,y) \in \rho\}$$
$$= \{(x,y) \in A \times B: \exists z \in B: x = z, (z,y) \in \rho\}$$
$$= \{(x,y) \in A \times B: (x,y) \in \rho\} = \rho$$

The statements are proved.