**Problem 1**

(a) Find the Laplace transform of the following:

(i) $5t^4 + 3t^3 - 3$ (ii) $3\sin 2t - 2\cos 5t - e^{-t}$

(b) Find the inverse Laplace transform of:

$$(i) \quad \frac{2s+3}{(s-3)(s-1)^2} \qquad (ii) \quad \frac{2s+1}{4s^2+1}$$

**Solution**

(a)

(i)

$$\mathcal{L}(5t^4 + 3t^3 - 3) = 5\mathcal{L}(t^4) + 3\mathcal{L}(t^3) - 3\mathcal{L}(1) = \frac{5!}{s^5} + \frac{3\cdot 3!}{s^4} - \frac{3}{s} = \frac{120}{s^5} + \frac{18}{s^4} - \frac{3}{s}$$

(ii)

$$\mathcal{L}(3\sin(2t) - 2\cos(5t) - e^{-t}) = 3\mathcal{L}(\sin(2t)) - 2\mathcal{L}(\cos(5t)) - \mathcal{L}(e^{-t})$$
$$= \frac{6}{s^2+4} - \frac{2s}{s^2+25} - \frac{1}{s+1}$$

(b)

(i)

$$\mathcal{L}^{-1}\left(\frac{2s+3}{(s-3)(s-1)^2}\right) = \mathcal{L}^{-1}\left(-\frac{9}{4(s-1)} - \frac{5}{2(s-1)^2} + \frac{9}{4(s-3)}\right)$$
$$= -\frac{9}{4}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + \frac{9}{4}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$
$$= -\frac{9}{4}e^t - \frac{5}{2}e^t t + \frac{9}{4}e^{3t}$$

(ii)

$$\mathcal{L}^{-1}\left(\frac{2s+1}{4s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{s}{s^2+\frac{1}{4}} + \frac{1}{4}\frac{1}{s^2+\frac{1}{4}}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{s}{s^2+\frac{1}{4}}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right)$$
$$= \frac{1}{2}\cos\left(\frac{t}{2}\right) + \frac{1}{2}\sin\left(\frac{t}{2}\right)$$

**Problem 2**

Use the method of Laplace transforms to solve the differential equation:

$$2\frac{dx}{dt} + 3x = 4\cos 2t, \qquad x(0) = 1$$

**Solution**

Let $\mathcal{L}\big(x(t)\big) = F(s)$

Let's apply Laplace transform to this equation.

$$\mathcal{L}\left(2\frac{dx}{dt} + 3x\right) = \mathcal{L}(4\cos(2t))$$

$$2\mathcal{L}\left(\frac{dx}{dt}\right) + 3\mathcal{L}(x) = 4\mathcal{L}(\cos(2t))$$

$$2\big(sF(s) - x(0)\big) + 3F(s) = \frac{4s}{s^2 + 4}$$

$$F(s)(2s + 3) = 2 + \frac{4s}{s^2 + 4}$$

$$F(s) = \frac{2s^2 + 4s + 8}{(s^2 + 1)(2s + 3)}$$

To find solution let's apply inverse Laplace transform.

$$x(t) = \mathcal{L}^{-1}\big(F(s)\big) = \mathcal{L}^{-1}\left(\frac{s^2 + 4s + 8}{(s^2 + 1)(2s + 3)}\right) = e^{-\frac{3}{2}t} + 2\sin(t)$$

**Problem 3**

(a) Find the general solution of the homogeneous equation:

$$3\frac{dx}{dt} - 2x = 0$$

(b) Find the particular solution in each of the following cases:

$$(i)\quad 3\frac{dx}{dt} - 2x = -5, \qquad (ii)\quad 3\frac{dx}{dt} - 2x = 2e^{-t}$$

(c) Hence find the solution of

$$3\frac{dx}{dt} - 2x = 2e^{-t},\ \ x(0) = 0$$

**Solution**

(a)

$$3\frac{dx}{dt} - 2x = 0$$

$$\frac{3dx}{x} = 2dt$$

$$\ln x = \frac{2}{3}t + \ln c$$

$$x(t) = ce^{\frac{2}{3}t}$$

(b) (i)

$$3\frac{dx}{dt} - 2x = -5$$

I will look for particular solution in form of constant

$$x(t) = c_0$$

$$-2c_0 = -5$$

$$c_0 = \frac{5}{2}$$

So, $x(t) = \frac{5}{2}$ is particular integral of the equation.

(ii)

I will find partial solution in form of

$$x(t) = ce^{-t}$$

After substituting we will get:

$$-3ce^{-t} - 2ce^{-t} = 2e^{-t}$$

$$5c = -2$$

$$c = -\frac{2}{5}$$

$$x(t) = -\frac{2}{5}e^{-t}$$

(c)

General solution of

$$3\frac{dx}{dt} - 2x = 2e^{t}$$

is sum of general solution of homogeneous equations and particular solution of non-homogeneous equation.

So,

$$x(t) = ce^{\frac{2}{3}t} - \frac{2}{5}e^{-t}$$

Let's find c from initial condition $x(0) = 0$

$$x(0) = c - \frac{2}{5} = 0$$

So, $c = \frac{2}{5}$.

So, solution is

$$x(t) = \frac{2}{5}\left(e^{\frac{2}{3}t} - e^{-t}\right)$$

**Problem 4**

Use the method of integration by parts to evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (3+t)\sin 2t\,dt$$

**Solution**

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} (3+t)\sin(2t)\,dt = (integration\ by\ parts)$$

$$= \frac{(3+t)(-\cos(2t))}{2}\Bigg|_{t=\left(-\frac{\pi}{3}\right)}^{t=\frac{\pi}{4}} - \int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{-\cos(2t)}{2}\,dt$$

$$= \frac{1}{2}\left(\left(3+\frac{\pi}{4}\right)(0) - \left(3-\frac{\pi}{3}\right)\left(\frac{1}{2}\right)\right) + \frac{1}{2}\frac{\sin(2t)}{2}\Bigg|_{t=-\frac{\pi}{3}}^{t=\frac{\pi}{4}}$$

$$= -\frac{1}{4}\left(3-\frac{\pi}{3}\right) + \frac{1}{4}\left(1+\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} + \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$