**Problem**

Find the optimal control

$$J(x,u) = \int_0^4 (u^2 + x)dt \to min$$

$$\dot{x} = u$$

$$x(0) = 0; |u| \leq 1$$

**Solution**

Let's rewrite this problem as a maximum problem.

$$-\int_0^4 (u^2 + x)dt \to \max$$

Pontryagin function:

$$H = -\lambda_0(u^2 + x)dt + \psi u$$

Adjoint system:

$$\dot{\psi} = -\frac{\partial H}{\partial x} = \lambda_0$$

The transversality condition:

$$\psi(4) = \frac{\partial \Phi_0}{\partial x(1)} = 0$$

Let's examine the case $\lambda_0 = 0$. Then $\dot{\psi} = 0$ implies $\psi = const$. Transversality condition implies $\psi = 0$. We get $\lambda_0 = \phi = 0$ that contradicts condition of the theorem. So the problem has no singular solutions.

Let's set $\lambda_0 = 1$. Then

$$H = \psi u - u^2 - x \to max$$

$$\dot{\psi} = 1; \psi(4) = 0$$

H is the quadratic function of u. Vertex of the parabola may be found from

$$\frac{\partial H}{\partial u} = \psi - 2u = 0$$

If the vertex is inside control interval $[-1;1]$ then it is maximum point. Otherwise H reaches its maximum on the left or right boundaries of the interval. So

$$u^*(t) = \begin{cases} sgn\,\psi(t), |\psi(t)| > 2 \\ \dfrac{\psi(t)}{2}, |\psi(t)| \le 2 \end{cases}$$

Optimal solution depends on $\psi(t)$. Solving the adjoint system we receive

$\psi(t) = t - 4$. So

$$u^*(t) = \begin{cases} -1; 0 \le t \le 2 \\ \dfrac{t-4}{2}; 2 \le t \le 4 \end{cases}$$

Let's determine the phase trajectory corresponding to optimal control.

$$\dot{x} = u^*(t) = \begin{cases} -1; 0 \le t \le 2 \\ \dfrac{t-4}{2}; 2 \le t \le 4 \end{cases}$$

$$x(t) = \begin{cases} -t + c_1;\ \ 0 \le t \le 2 \\ \dfrac{t^2}{4} - 2t + c_2;\ \ 2 \le t \le 4 \end{cases}$$

For the part of the trajectory $t \in [0;2]$ constant $c_1$ may be found from initial condition $x(0) = 0 \Rightarrow c_1 = 0$. For the part $t \in [2;4]$ let's use continuity of the trajectory at point $t = 2$:

$$\lim_{t \to 2-} x(t) = \lim_{t \to 2+} x(t)$$

Solving the equation we get $c_2 = 1$. So, finally,

$$u^*(t) = \begin{cases} -1; 0 \le t \le 2 \\ \dfrac{t-4}{2}; 2 \le t \le 4 \end{cases}; \quad x^*(t) = \begin{cases} -t;\ \ 0 \le t \le 2 \\ \dfrac{t^2}{4} - 2t + 1;\ \ 2 \le t \le 4 \end{cases}$$