Problem

Let $G = (V, E)$ be a directed graph and let $A = \left(a_{ij}\right)_{i,j=1,\dots,n}$ be a vertex adjacency matrix, i.e., $a_{ij} = 1$ if and only if $(i, j) \in A$.

Consider the following algorithm

for k=1 to n

for j=1 to n

for i=1 to n

$$a_{ij} = \max\{a_{ij}, \min(a_{ik}, a_{kj})\}$$

Show that at the end of the algorithm $a_{ij} = 1$ if and only if there exists a directed path from i to j in G.

Solution

Let A be a set such that $A = \{a_1, a_2, \dots, a_n\}$ and R be a relation defined on A. Considering a general path $x_1; x_2; \dots; x_m$ in R. Let's call vertices $x_2, \dots, x_{n-1}$ the interior vertices. Now, we define a matrix W_k $(1 \le k \le n)$ as follows. W_k has a 1 in position $(i; j)$ if and only if there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1; a_2; \dots; a_{k-1}\}$.

Let us suppose $W_k = \left[p_{ij}\right]$ and $W_{k-1} = \left[s_{ij}\right]$. If $p_{ij} = 1$, then there exists a path from a_i to a_j in R whose interior vertices are from $\{a_1, a_2, \dots a_k\}$. If the vertex a_k is not an interior vertex in this path, then all the interior vertices are from $\{a_1; a_2; \dots; a_{k-1}\} \Rightarrow s_{ij} = 1$. If the vertex a_k is an interior vertex in the path, then there will be a subpath from a_i to a_j and another subpath from a_k to a_j. All the interior vertices in the path from a_i to a_j are distinct. So a_k appears in the path only once, and hence all the interior vertices in the two subpaths mentioned earlier came from the set $\{a_1, \dots, a_{k-1}\}$. This means that $s_{ik} = 1$ and $s_{kj} = 1$.

Thus $p_{ij} = 1$ only under two conditions. If and only if either $s_{ij} = 1$, OR $s_{ik} = 1$ and $s_{kj} = 1$. So if W_{k-1} has 1 in the position $(i; j)$, so will W_k. A new 1 can be added at position $(i; j)$ in W_k if and only if column k in W_{k-1} has a 1 at position i and row k of W_{k-1} has a 1 at position j.

The algorithm builds matrix W_n. But W_n contains 1 on if there exists path from a_i to a_j. So, the statement is proved.