Problem 1

Diagonalize the matrix

$$A = \begin{pmatrix} 11 & 2 & -8 \\ 2 & 2 & 10 \\ -8 & 10 & 5 \end{pmatrix}$$

using the orthogonal transformation. Find the transformation matrix.

Solution

Matrix A is symmetric $A = A^T$. Every symmetric matrix represents some selfadjoint operator in orthonormal basis e.

Let's find eigenvalues of A. To do this we need to solve characteristic equation.

$$|A - \lambda I| = \begin{vmatrix} 11-\lambda & 2 & -8 \\ 2 & 2-\lambda & 10 \\ -8 & 10 & 5-\lambda \end{vmatrix} = -(\lambda - 18)(\lambda - 9)(\lambda + 9) = 0$$

Numbers $\lambda_1 = 18; \lambda_2 = 9; \lambda_3 = -9$ are roots of the characteristic equation.

To build a basis from eigenvectors we need to find fundamental system of solutions of system of linear equations $(A - \lambda E)X = 0$ for each eigenvalue λ.

For $\lambda_1 = 18$ the system is

$$\begin{cases} -7x_1 + 2x_2 - 8x_3 = 0 \\ 2x_1 - 16x_2 + 10x_3 = 0 \\ -9x_1 + 10x_2 - 13x_3 = 0 \end{cases}$$

Rank of the matrix equals 2, so fundamental system of solutions consists of $n - r = 3 - 2 = 1$ vector. General solution of the system

$$X = \begin{pmatrix} -\alpha \\ \frac{\alpha}{2} \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ \frac{1}{2} \\ 1 \end{pmatrix}, \forall \alpha$$

where $\begin{pmatrix} -1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$ is the fundamental solution of the system. Normalizing this vector: $f_1 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

For $\lambda_2 = 9$ the system is

$$\begin{cases} 2x_1 + 2x_2 - 8x_3 = 0 \\ 2x_1 - 7x_2 + 10x_3 = 0 \\ -8x_1 + 10x_2 - 4x_3 = 0 \end{cases}$$

Rank of the system is 2, so fundamental solution consists of $n - r = 3 - 2 = 1$ vector. General solution of the system is

$$X = \begin{pmatrix} 2\alpha \\ 2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \forall \alpha$$

where $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is the fundamental solution of the system. Normalizing this vector: $f_2 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$.

For $\lambda_3 = -9$ the system is

$$\begin{cases} 20x_1 + 2x_2 - 8x_3 = 0 \\ 2x_1 + 11x_2 + 10x_3 = 0 \\ -8x_1 + 10x_2 + 14x_3 = 0 \end{cases}$$

Rank of the matrix equals 2 so fundamental solution consists of 1 vector. Solution of the system is $X = \begin{pmatrix} \frac{\alpha}{2} \\ -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} \frac{1}{2} \\ -1 \\ 1 \end{pmatrix}$. $\begin{pmatrix} \frac{1}{2} \\ -1 \\ 1 \end{pmatrix}$ is fundamental solution of the system. Normalizing this vector $f_3 = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$.

Since eigenvectors of the selfadjoint operator corresponding to different eigenvalues are orthogonal then vectors we found

$f_1 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$; $f_2 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$; $f_3 = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$ form the orthogonal basis that consists of eigenvectors. Matrix of linear operator in this basis equals

$$A_f = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

Transformation matrix from e to f equals

$$U = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Since U is the transformation matrix from orthogonal basis to orthogonal basis so it is orthogonal:

$$A_f = U^{-1}AU = U^T AU$$