Stability of Linear Systems

Routh–Hurwitz stability criterion

Definitions

Let $$\dot{x} = f(x, t) \qquad (1)$$

be a linear system defined in the region $\Omega = [a; \infty) \times D$ where $D \subset R^n$.

Suppose there exists a unique solution of Cauchy problem for any initial conditions.

Def: $x^*(t)$ is called a **stable solution** of system (1) if the following conditions are satisfied:

1) $x^*(t)$ is defined on $[a;\ \infty)$.
2) $\forall \varepsilon > 0\ \ \forall t_0 \geq a\ \ \exists \delta > 0\ \forall x_0\colon \|x_0 - x^*(t_0)\| < \delta \Rightarrow \forall t \geq t_0\ \|x(t) - x^*(t)\| < \varepsilon$ where $x(t)$ satisfies the initial condition $x(t_0) = x_0$ and is defined on $[t_0; \infty)$

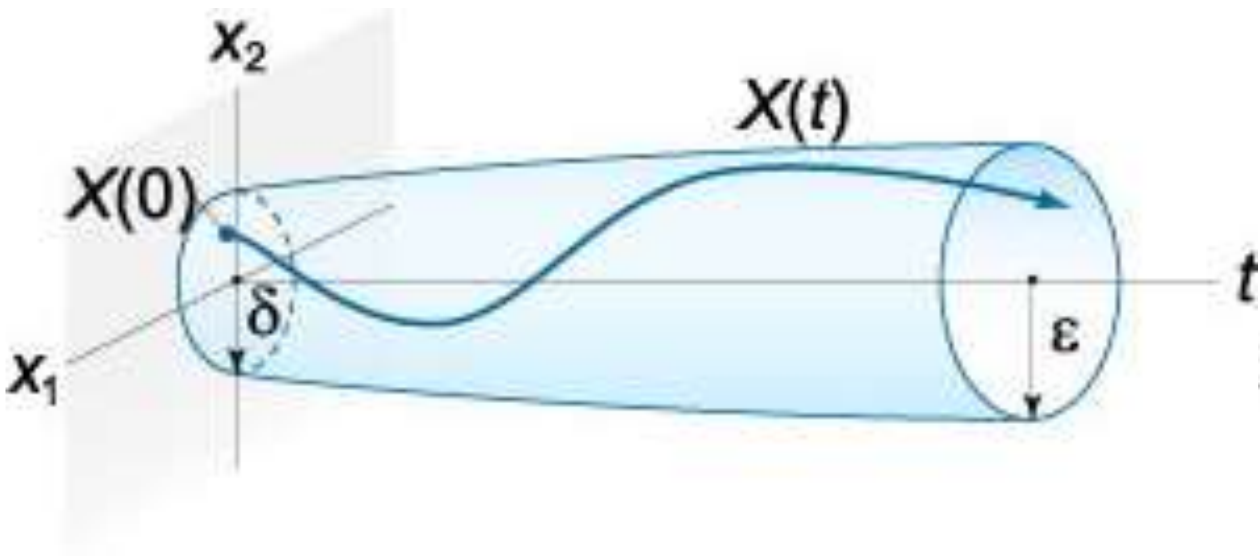
In case

$$\lim_{t \to \infty} \|x(t) - x^*(t)\| = 0$$

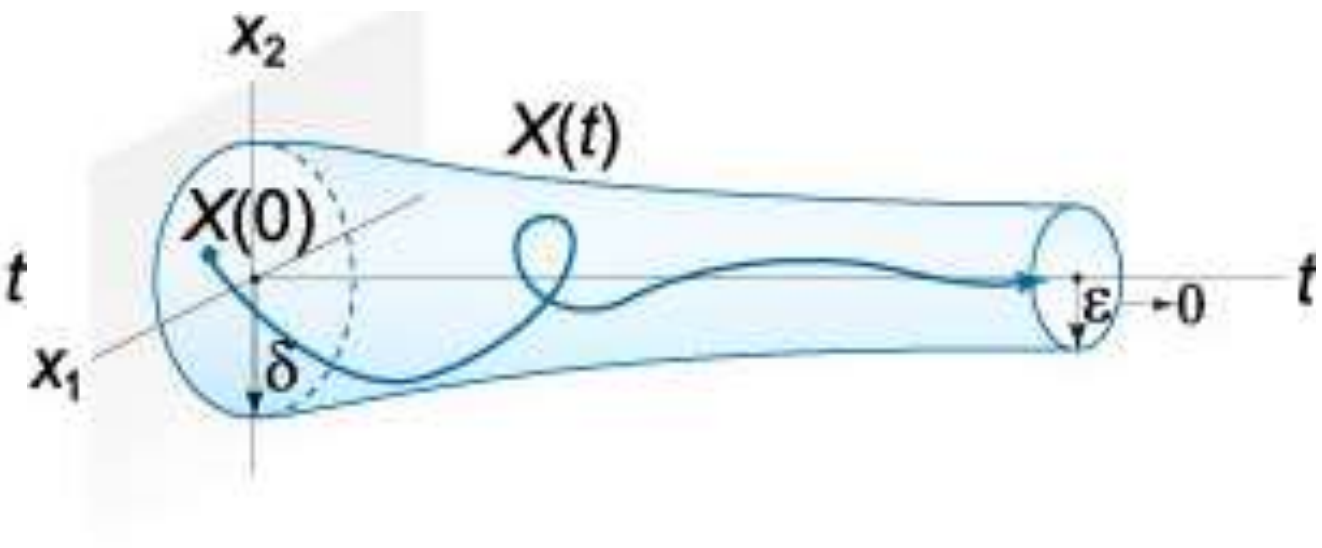
$x^*(t)$ is called **asymptotically stable solution**.

Graphical interpretation

Stability in the sense of Lyapunov



Asymptotic stability



Example

Problem 1

$y'(t) = ky(t),\ k = const,\ y(0) = y_0$

For each k determine whether the solution is stable.

Solution:

The solution is $y(t) = y_0 e^{kt}$.

1) $k < 0$: $\lim_{t\to\infty} \left| y_1 e^{kt} - y_0 e^{kt} \right| = \lim_{t\to\infty} e^{kt} |y_1 - y_0| = 0$ as $e^{kt} \to 0$ when $t \to \infty$.
2) $k = 0$: assuming $\delta = \varepsilon$ we get $|y_1 - y_0| < \delta = \varepsilon\ \forall t \geq 0$.
3) $k > 0$: $\lim_{t\to\infty} \left| y_1 e^{kt} - y_0 e^{kt} \right| = \lim_{t\to\infty} e^{kt} |y_1 - y_0| = \infty$ as $e^{kt} \to \infty$.

So the solution is asymptotically stable when $k < 0$, stable when $k = 0$ and unstable when $k > 0$.

Criteria for Linear Systems

Theorem: Consider the linear system $\dot{y}(t) = Ay(t)$, A – constant matrix. (2)

Let λ_i be eigenvalues of matrix A. In other words λ_i are the roots of characteristic polynomial $\det(A - \lambda I) = 0$.

1) The solution of the system (2) is asymptotically stable if and only if all eigenvalues λ_i satisfy $Re(\lambda_i) < 0$.
2) The solution of (2) is stable if and only if for each eigenvalue λ_i of A holds $Re(\lambda_i) \leq 0$ and if $Re(\lambda_i) = 0$ then the algebraic multiplicity of λ_i equals to its geometric multiplicity.
3) The solution of (2) is unstable iff $\exists i \; Re(\lambda_i) > 0$ or $Re(\lambda_i) = 0$ and geometric multiplicity is less than algebraic multiplicity of λ_i.

Example

Problem 2

Determine whether the following system is stable:

$$\begin{cases} y_1' = y_1 + 4y_2 \\ y_2' = 2y_1 + 3y_2 \end{cases}$$

Solution:

$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Let's find the eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$$

The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$.

Since $Re(\lambda_1) = Re(5) = 5 > 0$ the solution is unstable.

Routh-Hurwitz criterion

Theorem: Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be a characteristic polynomial of a matrix A, $a_0 > 0$, $a_i \in \mathbb{R}$. Matrix $H = \{h_{i,j}\}_{i,j=1}^{n}$ defined as

$$h_{ij} = \begin{cases} a_{2i-j}, 0 \leq 2i - j \leq n \\ 0, otherwise \end{cases}$$

$$H = \begin{pmatrix} a_1 & a_0 & \dots & 0 \\ a_3 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n \end{pmatrix}$$

is called Hurwitz matrix. Then the system (2) is asymptotically stable if and only if all the leading principal minors of H are positive.

Example

Problem 3

Determine the values of k for which the following system

$$\begin{cases} y_1' = 2y_1 - 7y_2 \\ y_2' = 4y_1 + ky_2 \end{cases}$$

is asymptotically stable.

Solution:

$$P(\lambda) = \det(A - \lambda I) = (2-\lambda)(k-\lambda) + 28 = \lambda^2 - (k+2)\lambda + (2k+28)$$

$$H = \begin{pmatrix} -(k+2) & 1 \\ 0 & 2k+28 \end{pmatrix}$$

Leading principal minors of H are $-(k+2) > 0$ and $-(k+2)(2k+28) > 0$

So the system is asymptotically stable when $k \in (-14, -2)$.

Have any questions?

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