Integration of trigonometric functions

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Rational functions

Def: function $R(x,y)$ is called rational function of two variables in case it can be written in the form $R(x,y)=\frac{P(x,y)}{Q(x,y)}$ where $P(x,y)$ and $Q(x,y)$ are polynomials.

Rational functions of 2 variables:

$$R(x,y)=\frac{x^2+xy+y^2}{3y^2-x} \qquad R(x,y)=\frac{x}{y}$$

Rational functions of $\sin(x)$ and $\cos(x)$:

$$R(\sin x,\cos x)=\frac{\sin x}{\cos x}=\tan x$$

$$R(\sin x,\cos x)=\frac{3\sin^2 x-\sin^4 x\cos x}{\cos x-2\sin x+3}$$

U-Substitution

Universal substitution reduces trigonometric integral to rational

$$t = \tan\frac{x}{2} \qquad x = 2\arctan t \qquad dx = \frac{2}{1+t^2}dt$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2};$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\cdot\frac{2}{1+t^2}dt$$

Example of using U-Substitution

$$\int \frac{dx}{3 + \sin x + \cos x}$$

Substitution:

$$t = \tan\frac{x}{2} \qquad dx = \frac{2dt}{1+t^2} \qquad \sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Result:

$$\int \frac{dx}{3 + \sin x + \cos x} = \int \frac{2dt}{(1+t^2)\left(3 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}\right)} = \int \frac{dt}{t^2 + t + 2}$$

$$= \int \frac{d\left(t + \frac{1}{2}\right)}{\left(t + \frac{1}{2}\right)^2 + \frac{7}{4}} = \frac{2}{\sqrt{7}} \arctan \frac{t + \frac{1}{2}}{\sqrt{7}/2} + C = \frac{2}{\sqrt{7}} \arctan \frac{1 + 2\tan\frac{x}{2}}{\sqrt{7}} + C$$

Simpler ways

U-Substitution often leads to long calculations ☹

If $R(\cdot,\cdot)$ has some useful properties, we can use another substitutions to simplify the integral:

1) $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ then use $t = \cos x$

2) $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ then use $t = \sin x$

3) $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ then use $t = \tan x$

4) $R(\sin x, \cos x) = R(\tan x)$ then use $t = \tan x$

Example

$$\int \frac{dx}{1+\sin^2 x}$$

Substitution:

$R(\sin x, \cos x) = \frac{1}{1+\sin^2 x}$ $\quad R(-\sin x, -\cos x) = \frac{1}{1+(-\sin x)^2} = \frac{1}{1+\sin^2 x} =$ $R(\sin x, \cos x)$, so we use the substitution $t = \tan x$.

$$x = \arctan t \quad dx = \frac{dt}{1+t^2} \quad \sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2}$$

Result:

$$\int \frac{dx}{1+\sin^2 x} = \int \frac{dt}{(1+t^2)\left(1+\frac{t^2}{1+t^2}\right)} = \int \frac{dt}{2t^2+1}$$

$$= \frac{1}{\sqrt{2}} \arctan \sqrt{2}t + C = \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\tan x\right) + C$$

Non-integrable functions

Integrals of some functions cannot be expressed as analytic functions.

Some basic examples of such trigonometric functions:

$$\sin x^2, \cos x^2, \tan x^2$$

$$\tan\sqrt{x}$$

$$\sin\left(\frac{1}{x}\right), \cos\left(\frac{1}{x}\right), \tan\left(\frac{1}{x}\right)$$

$$\frac{\sin x}{x}, \frac{\cos x}{x}, \frac{\tan x}{x}$$

$$x \tan x$$

$$\sqrt{\sin x}, \sqrt{\cos x}$$

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