Compact Metric Spaces

Topological Space

Let X be an arbitrary set. Set τ of subsets of X is called **topology**, in case it satisfies such conditions:

1) $\emptyset, X \in \tau$
2) τ is closed under arbitrary union:

$$\forall i \in I \; G_i \in \tau \Rightarrow \bigcup_{i \in I} G_i \in \tau$$

3) τ is closed under finite intersection:

$$G_i \in \tau, i = 1, \dots, n \Rightarrow \bigcap_{i=1}^{n} G_i \in \tau$$

Elements of τ are called **open sets** in the **topological space** (X, τ).

Examples of Topological Spaces

Example 1 (space with trivial topology):

$X = \{1, 2, 3\}$, $\tau = \{\emptyset, \{1, 2, 3\}\}$, (X, τ) – topological space.

Example 2 (Sierpinski space):

$X = \{0, 1\}$, $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$, (X, τ) – topological space (Sierpinski space).

All possible results of unions: $\emptyset$, $\{1\}$ and $\{0, 1\}$.

All possible results of finite intersections: $\emptyset$, $\{1\}$ and $\{0, 1\}$.

Example 3 (cofinite topology on $\mathbb{R}$):

$X = \mathbb{R}$, $\tau = \{A \subseteq \mathbb{R} \mid A = \emptyset \ or \ X \backslash A \ is \ finite\}$, $(\mathbb{R}, \tau)$ – topological space.

Any union of sets from τ: $\emptyset$, $\mathbb{R}$ or a set of all elements of $\mathbb{R}$ except a finite set of elements.

Any finite intersection: $\emptyset$, $\mathbb{R}$ or a set of all elements of $\mathbb{R}$ except a finite set of elements.

Metric space

Let X be an arbitrary set. The map $\rho: X \times X \to \mathbb{R}^+$ is called **metric** in case for all $x, y \in X$ such properties hold:

1) Non-negativity: $\rho(x, y) \geq 0$
2) Coincidence: $\rho(x, y) = 0 \Leftrightarrow x = y$
3) Symmetry: $\rho(x, y) = \rho(y, x)$
4) Triangle inequality: $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

The pair (X, ρ) is called **metric space**.

Examples of metric spaces

Discrete metric:

$$\rho(x, y) = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases}$$

Euclidean metric on $\mathbb{R}^n$:

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Maximum metric on the space of continuous functions $C([a, b])$:

$$\max_{t \in [a,b]} |x(t) - y(t)|$$

Open sets in metric spaces

Consider the metric space (X, ρ). The set $S(x_0, \varepsilon) = \{x \in X \mid \rho(x, x_0) < \varepsilon\}$ is called open ball with the center x_0 and radius ε.

The set $G \subset X$ in the metric space is called open in case $\forall x \in G\ \exists r > 0: S(x, r) \subset G$.

So the metric on X induces the topology on X (set of all open subsets of X).

Thus each metric space is a topological space.

Compactness

Topological space (X, τ) is called **compact**, in case each open cover of X contains finite subcover:

$$X = \bigcup_{i \in I} O_i$$

(O_i are open sets – $O_i \in \tau$), then

$$X = \bigcup_{j=1}^{n} O_{i_j}$$

It is also called Borel-Lebesgue condition.

Sequential compactness

Convergence in topological space (X, τ) is defined in such a way:

$x_n \to x_0$ if for every $U \in \tau$, such that $x_0 \in U$ (U is open neighborhood of x_0) there exists N such that $\forall n \geq N$ $x_n \in U$.

If the topology τ was induced from the metric ρ, then convergence $x_n \to x_0$ is determined by relation $\rho(x_n, x_0) \to 0$.

The topological space (X, τ) is called **sequentially compact**, in case any sequence $\{x_n\} \in X$ contains convergent subsequence $x_{n_k} \to x_0$.

Limit point compactness

The point $x_0 \in X$ in the topological space is called a **limit point** of the set $S \subset X$ in case each open neighborhood U of point x_0 contains the point from S (different from x_0).

The space (X, τ) is called **limit point compact**, in case any infinite subset of X contains a limit point.

Definitions

Metric space (X, ρ) is called **complete** in case each Cauchy sequence x_n

$$\forall \varepsilon > 0\ \exists N: \forall\, n, m \geq N\ \rho(x_n, x_m) < \varepsilon$$

is convergent to some element $x_0 \in X$.

Let $A \subset X$. The set $B \subset X$ is called ε - net for the set A if

$$\forall x \in A\ \exists y \in B: \rho(x, y) < \varepsilon$$

The set A is called **totally bounded**, in case for each $\varepsilon > 0$ there exists finite ε-net.

Theorem

Let (x, ρ) be a metric space. Then the following 4 statements are equivalent:

1) (X, ρ) is **compact**.
2) (X, ρ) is **complete** and **totally bounded**.
3) (X, ρ) is **sequentially compact**.
4) (X, ρ) is **limit point compact**.

Questions?

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