Characteristic function of a random variable

Definition

Def: Let ξ be a random variable. Then the function $\varphi_\xi : \mathbb{R} \to \mathbb{C}$ defined as

$$\varphi_\xi(t) = Ee^{it\xi} = E(\cos \xi t + i \sin \xi t) = E \cos \xi t + iE \sin \xi t$$

is the characteristic function of ξ.

Alternative definition

Def: If $F_\xi(x)$ is the cumulative distribution function of ξ then

$$\varphi_\xi(t) = \int_{\mathbb{R}} e^{itx} \, dF_\xi(x)$$

Discrete and continuous variables

If ξ is a discrete random variable with a set of values $\{x_k\}$ then

$$\varphi_\xi(t) = \sum_k e^{itx_k} P\{\xi = x_k\}$$

If ξ is a continuous random variable with pdf $f_\xi(x)$ then

$$\varphi_\xi(t) = \int_{\mathbb{R}} e^{itx} f_\xi(x) dx$$

If we know the characteristic function $\varphi_\xi(t)$ of ξ we can apply the Fourier transform and find the pdf $f_\xi(x)$:

$$f_\xi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx}\, \varphi_\xi(t) dt$$

Properties

1) $\forall t \in \mathbb{R}\ |\varphi(t)| \leq 1, \varphi(0) = 1$
2) $\varphi(t)$ is uniformly continuous
3) If $\eta = a\xi + b$ (a and b - constants) then $\varphi_\eta(t) = e^{itb}\varphi_\xi(at)$
4) If $\xi_1, \dots, \xi_n$ are independent then

$$\varphi_{\xi_1+\cdots+\xi_n}(t) = \prod_{k=1}^{n} \varphi_{\xi_k}(t)$$

5) $\varphi_\xi(-t) = \varphi_{-\xi}(t) = \overline{\varphi_\xi}(t)$
6) If $m_n = E\xi^n < \infty$ then $\varphi^{(k)}(0) = i^k E\xi^k$
7) ξ is symmetric random variable if and only if $\varphi_\xi(t)$ is a real-valued function.

Characteristic function examples

Distribution	**Characteristic function**
Constant ($P(\xi = c) = 1$)	e^{itc}
Uniform $U(a, b)$	$\dfrac{e^{itb} - e^{ita}}{it(b-a)}$
Poisson $Pois(\lambda)$	$e^{\lambda(e^{it}-1)}$
Bernoulli $Bern(p)$	$1 - p + pe^{it}$
Binomial $B(n, p)$	$(1 - p + pe^{it})^n$
Normal $N(\mu, \sigma^2)$	$e^{it\mu - \frac{1}{2}\sigma^2 t^2}$
Exponential $Exp(\lambda)$	$(1 - it\lambda^{-1})^{-1}$
Chi-squared X_k^2	$(1 - 2it)^{-\frac{k}{2}}$
Gamma $\Gamma(k, \theta)$	$(1 - it\theta)^{-k}$

Examples of usage

Problem 1

Suppose ξ_1 and ξ_2 – independent normally distributed random variables. Prove that $\xi_1 + \xi_2$ is normally distributed.

Proof:

$\xi_1 \sim N(m_1, \sigma_1^2)$, $\xi_2 \sim N(m_2, \sigma_2^2)$. The characteristic functions are:

$\varphi_1(t) = e^{itm_1 - \frac{1}{2}\sigma_1^2 t^2}$ and $\varphi_2(t) = e^{itm_2 - \frac{1}{2}\sigma_2^2 t^2}$.

Using the property (4):

$$\varphi_{\xi_1+\xi_2}(t) = \varphi_1(t)\varphi_2(t) = e^{itm_1 - \frac{1}{2}\sigma_1^2 t^2} e^{itm_2 - \frac{1}{2}\sigma_2^2 t^2} =$$

$= e^{it(m_1+m_2) - \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$. So $\xi_1 + \xi_2 \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

Examples of usage

Problem 2

Suppose ξ and η are independent identically distributed random variables. Prove that $\nu = \xi - \eta$ is symmetric.

Proof:

Let's use the property (7):

$$\varphi_\nu(t) = Ee^{it(\xi-\eta)} = Ee^{it\xi}\, e^{-it\eta} = |\xi \text{ and } \eta \text{ are independent}| =$$

$$= Ee^{it\xi}\, Ee^{-it\eta} = \varphi_\xi(t)\varphi_\eta(-t)$$

Since ξ and η are identically distributed:

$$\varphi_\xi(t)\varphi_\eta(-t) = \varphi_\xi(t)\varphi_\xi(-t) = |\text{using the property (5)}| =$$

$= \varphi_\xi(t)\bar{\varphi}_\xi(t) = \left|\varphi_\xi(t)\right|^2$ is real-valued. So ν is symmetric.

Have any questions?

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